

Pure Mathematics

Differential & Integral Calculus

3rd Secondary

Concepts Sheet



Differentiation and it's Applications

Derivatives of the trigonometric function

Function		Derivative
Sine function	$\sin x$	$\cos x$
Cosine function	$\cos x$	$-\sin x$
Tangent function	$\tan x$	$\sec^2 x$
Cotangent function	$\cot x$	$-\csc^2 x$
Secant function	$\sec x$	$\sec x \tan x$
Cosecant function	$\csc x$	$-\csc x \cot x$

Implicit Differentiation

Differentiating the implicit relation $f(x, y) = 0$ requires to differentiate both sides of the relation with respect to one of the two variables x or y according to the chain rule to get $\frac{dy}{dx}$ or $\frac{dx}{dy}$ respectively .

Parametric Differentiation

The curve given in the parametric form $y = f(t)$, $x = g(t)$,

then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

where f and g are two differentiable functions with respect to t

Higher - Derivatives of a functions

If $y = f(x)$ where f is a differentiable function with respect to x,then the derivatives starting from the second derivative (if found) are called the higher derivatives and they are denoted the symbol $\frac{d^2y}{dx^2}$ or y'' , the third derivative is denoted by the symbol y''' or $\frac{d^3y}{dx^3}$ and the n[th] derivatives is denoted by the symbol $y^{(n)}$ or $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$ where n is a positive integer number.

The two equations of the tangent and the normal to a curve

If m the slope of the tangent to the curve $y = f(x)$ at the point (x_1 , y_1) which lies on it, then: the equation of the tangent to the curve at point (x_1 , y_1) is:

$$y - y_1 = m(x - x_1)$$

the equation of the normal at point (x_1 , y_1) is:

$$y - y_1 = \frac{-1}{m}(x - x_1)$$

Related time Rates

If $y = f(x)$, x changes over time t , then y changes over time t.

i.e. y is a composite function of the time t and $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

and this relation relates the time rate of change x with the time rate of change y .

The rate is positive if the variable increases with the increase of time.

The rate is negative if the variable decreases with the increase of time.



The Calculus of Exponential and Logarithmic Functions

The number e is known by the relation

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} \quad , \quad e = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$$

$$\lim_{x\to 0}\frac{a^{x}-1}{x} = \ln a \quad ,a>0 \qquad , \quad \lim_{x\to 0}\frac{\log_a(1+x)}{x} = \log_a e \quad ,a>0$$

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$$

The exponential function with the natural base is a function whose base is e where

$$f(x) = e^{x}, x \in R$$

The natural logarithmic function is a logarithmic function whose base is e where

$$f(x) = \ln x \text{ and } x \in R^{+}$$

Logarithmic differentiation:

The relation among the variables can be expressed in a logarithmic form by taking off the natural logarithm of its both sides and using the properties of logarithms to simplify the relation before doing the differentiation operations

Some properties of the natural logarithm.

If $\boldsymbol{x \in R^{+}, y \in R, a \in R^{+}-\{1\}}$***, then:***

(1) The form $\boldsymbol{y = \ln x}$ is equivalent to the form $\boldsymbol{e^{y} = x}$

(2) $e^{\ln x} = x$ **(3)** $\ln e = 1$ **(4)** $\ln 1 = 0$ **(5)** $\log_a x = \frac{\ln x}{\ln a}$

For each $\boldsymbol{x \in R^{+}, y \in R^{+}, n \in R}$***, then:***

(6) $\ln xy = \ln x + \ln y$ **(7)** $\ln\frac{x}{y} = \ln x - \ln y$

(8) $\ln x^{n} = n \ln x$ **(9)** $\log_a x \times \log_x a = 1$

Derivatives of exponential and logarithmic functions

Function	Derivative	Condition
e^x	e^x	$x \in R$
$e^{f(x)}$	$e^{f(x)} \cdot f'(x)$	f is differentiable
a^x	$a^x \ln a$	$a > 0 , a \neq 1$
$\ln \lvert x \rvert$	$\frac{1}{x}$	$x \neq 0$
$\ln \lvert f(x) \rvert$	$\frac{1}{f(x)} \cdot f'(x)$	f is differentiable, $f(x) \neq 0$
$\log_a x$	$\frac{1}{x \ln a}$	$a > 0 , a \neq 1$
$\log_a f(x)$	$\frac{f'(x)}{f(x) \ln a}$	$a > 0 , a \neq 1$

Integration of exponential and logarithmic functions

Function	Integration	Condition
e^x	$e^x + c$	$x \in R$
e^{kx}	$\frac{1}{k} e^{kx} + c$	$k \neq 0$
$e^{f(x)} \cdot f'(x)$	$e^{f(x)} + c$	f is differentiable
$\frac{1}{x}$	$\ln \lvert x \rvert + c$	$x \neq 0$
$\frac{1}{f(x)} \cdot f'(x)$	$\ln \lvert f(x) \rvert + c$	f is differentiable, $f(x) \neq 0$



Behavior of the Function and Curve Sketching

First derivative test for the monotony of the functions:

Theorem: let f be a differentiable function on the interval $]a , b[$:

1- If $f'(x) > 0$ for all the value of $x \in]a , b[$,then f is increasing on $]a , b[$.

2- If $f'(x) < 0$ for all the values of $x \in]a, b[$,then f is decreasing on $]a, b[$.

The critical point:

The continuous function f on $]a, b[$ has a critical point $(c, f(c))$, if $c \in]a, b[$, $f'(c) = 0$ or $f'(c)$ is not existed.

The local maximum and minimum values

If f is a continuous function whose domain is I and $c \in I$, then the function f has:

1- A local maximum value at c if an open interval is found $]a, b[\subset I$ containing c where $f(x) \leq f(c)$ for each $x \in]a, b[$.

2- A local minimum value at c if an open interval is found $]a, b[\subset I$ containing c where $f(x) \geq f(c)$ for each $x \in]a, b[$.

First derivative test of the maximum and minimum values:

If $(c, f(c))$ is a critical point for the continuous function f at c , and an open interval is found around c where:

1- $f'(x) > 0$ when $x < c$, $f'(x) < 0$ when $x > c$, then $f(c)$ is a local maximum value.

2- $f'(x) < 0$ when $x < c$, $f'(x) > 0$ when $x > c$, then $f(c)$ is a local minimum value.

Theorem: If f is differentiable on $]a, b[$ and the function f has a local maximum value or a local minimum value at $c \in]a, b[$, then $f'(c) = 0$ or $f'(c)$ does not exists

The extrema values of a function on a closed interval:

Theorem: If the function f is continuous on the interval $[a, b]$, then the function f has an absolute maximum value and absolute minimum value on the interval $[a, b]$.

The absolute maximum values and the absolute minimum values of a function on a closed interval:

If the function f is defined on the closed interval $[a, b]$ and $c \in [a, b]$ then,

(1) $f(c)$ is an absolute minimum value on the interval $[a, b]$ when $f(c) \leq f(x)$ for each $x \in [a, b]$

(2) $f(c)$ is an absolute maximum value on the interval $[a, b]$ when $f(c) \geq f(x)$ for each $x \in [a, b]$

Second derivative test of the local maximum and minimum values

Theorem: let the function f has a second derivative on an open interval containing C where $f'(c) = 0$

1- If $f''(c) < 0$ then $f(c)$ is a local maximum value.

2- If $f''(c) > 0$ then $f(c)$ is a local minimum value.

Convexity of curves

Let the function f be differentiable on the interval $]a, b[$, the curve of the function f is convex downwards if f' is increasing on this interval and is convex upwards if f' is decreasing on this interval.

Second derivative test for convexity of curves

Theorem: Let the function f be differentiable twice on the interval $]a, b[$, then:

1- If $f''(x) > 0$ for all the value of $x \in]a, b[$, then the curve is convex downwards on the interval $]a, b[$.

2- If $f''(x) < 0$ for all the value of $x \in]a, b[$, then the curve is convex upwards on the interval $]a, b[$.

Inflection point

If f is a continuous function on the open interval $]a, b[$, $c \in]a, b[$ and the curve of the function has a tangent at point $(c, f(c))$, then this point is called the inflection point to the curve of the function f, if the convexity of the curve of the function changes at this point from being convex downward to upward or vice versa.



Differential of function:

If the function f is differentiable on an open interval containing x, $y = f(x)$, then:
$dy = f'(x)\,dx$ where dy is differential of y and dx is differential of x.

Integration by substitution:

It is one of the methods to find the integration of the product of two functions. We can use it to change the given integration into know standard integration.
If $z = g(x)$ is a differentiable function, then: $\int f(g(x))\, g'(x)\, dx = \int f(z)\, dz$

Integration by parts:

it is one of the methods to find the integration of the product of two functions each of them is not a derivative to the other. If y and z are two differentiable functions on the interval I , then

$\int y\, d z = y z - \int z\, d y$

Table of fundamental integrations (standard)

$\int x^n\, dx = \frac{x^{n+1}}{n+1} + c \quad , n \in R - \{1\}$	$\int sec\, x\ tan\, x\, dx = sec\, x + c$ $, x \neq \frac{2n+1}{2}\pi \quad , n \in Z$
$\int sin\, x\, dx = -cos\, x + c$	$\int csc\, x\, cot\, x\, dx = -csc\, x + c$ $x \neq n\pi \quad , n \in Z$
$\int cos\, x\, dx = sin\, x + c$	$\int e^x\, dx = e^x + c$
$\int sec^2 x\, dx = tan\, x + c$ $, x \neq \frac{2n+1}{2}\pi \quad , n \in Z$	
$\int csc^2 x\, dx = -cot\, x + c$ $, x \neq n\pi \quad , n \in Z$	$\int \frac{1}{x}\, dx = ln\,\lvert x\rvert + c \quad , x \neq 0$

⇒ Adding a constant (b) to the independent variable does not affect the formula of integration.

⇒As multiplying the variable x by the coefficient (a), the integration keeps its previous formula, but it divides this coefficient.

Definite integration:

Theorem:

If the function f is continuous on the interval $[a\, , b]$ and F is any antiderivative to the function f on the same interval, then $\int_a^b f(x) d\, x = F(b) - F(a)$

Properties of the definite integral:

If the function f is continuous on the interval $[\, a\, , b\,]$, $c \in\,]\, a\, , b[$, then:

1- $\int_b^a f(x) d\, x = -\int_a^b f(x) d\, x$

2- $\int_a^a f(x)d\,x = 0$

3- $\int_a^b f(x)d\,x = \int_a^c f(x)d\,x + \int_c^b f(x)d\,x$ for each $c \in [a, b]$

4- $\int_{-a}^a f(x)d\,x = 0$ (If f in an odd function)

5- $\int_{-a}^a f(x)d\,x = 2\int_0^a f(x)d\,x$ (If f in an even function)

Areas of the plane:

⇒The area of a region bounded by the curve of the continuous function f on the interval $[a, b]$, the $x-$ axis and the two straight lines: $x = a$ and $x = b$ where $f(x) \geq 0$ is:

$$A = \int_a^b f(x)\,d\,x$$

⇒The area of a region bounded by the curve of the two continuities functions f and g on the interval $[a, b]$ and the two straight lines: $x = a$ and $x = b$ where $f(x) \geq g(x) > 0$ is:

$$A = \int_a^b |f(x) - g(x)|d\,x$$

Volumes of revolution:

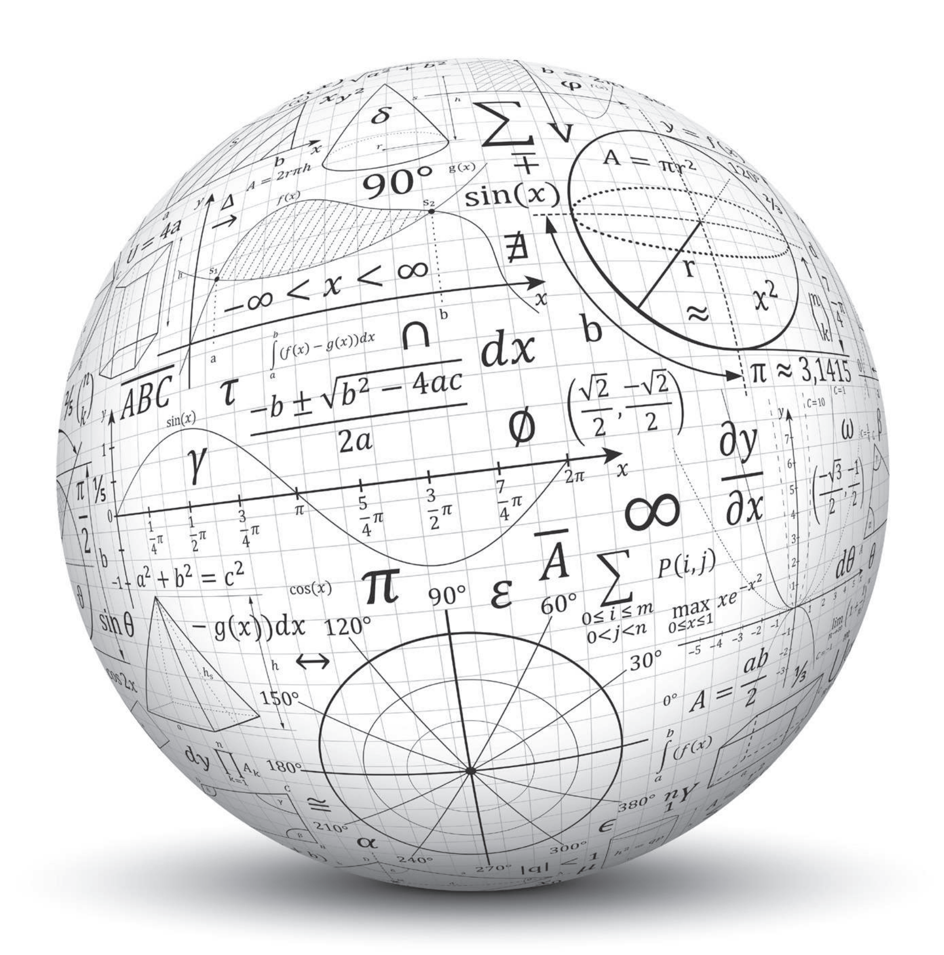
The solid of revolution is generated by revolving a plane region a complete revolution about a straight line is called the axis of revolution.

⇒ The volume of a solid generated by revolving a region bounded by the curve of the continuous function f on the interval $[a, b]$, the x-axis and the two straight lines $x = a$ and $x = b$ a complete revolution about the x-axis where $f(x) \geq 0$ is :

$$V = \pi \int_a^b (f(x))^2 d\,x$$

⇒The volume of a solid generated by revolving the region bounded by the two continuous functions f and g on the interval $[a, b]$, the x-axis and the two straight lines $x = a$ and $x = b$ a complete revolution about the x-axis where $f(x) \geq g(x) > 0$ is :

$$V = \pi \int_a^b |(f(x))^2 - (g(x))^2|\,d\,x$$



Mathématiques Pures

Calcul différentiel et intégral

Résumé de concepts

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